Technical Notes

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Hydrodynamic Solution for Incompressible Flow as Influenced by Gravitation

W. L. Chow,* T. Han,† and C. Wu†
University of Illinois at Urbana-Champaign
Urbana, Ill.

IN many simple fluid mechanical problems, the effect of the gravitational field has not been properly explored. Adequate solutions to problems such as the flow past a weir or a sluice gate where the effect of gravitation predominates have not been established. Approximate methods for some problems were recently developed by Mark,[1] Woronetz,[2] Benjamin,[3] and Gurevich.[4,5] Recent interest in the numerical solution of the hodograph equation[6] leads to the finding that this type of problem may be dealt with effectively through hodograph transformations. An example of treating one of these problems is presented here.

For a heavy, ideal incompressible fluid discharging into the atmosphere as shown in Fig. 1, it is desired for a specific uniform approaching flow and α to find the compatible downstream flow conditions, including the height of the opening h_c and the subsequent jet boundary.

From conservational principles, it can be shown that V_D/V_A satisfies a cubic algebraic equation given by

$$\left(\frac{V_D}{V_A}\right)^3-\left(C_{p_{A_a}}+1+\frac{2}{Fr_A^2}\right)\frac{V_D}{V_A}+\frac{2}{Fr_A^2}=0 \qquad (1)$$

where

$$C_{p_{Aa}}=\frac{p_a-p_{\mathrm{atm}}}{\rho V_A^2/2}$$

and F_{rA} is the Froude number defined by

$$Fr_A=V_A/(gh_A)^{1/2}$$

When $C_{p_{Aa}}$ vanishes, V_D/V_A is given by

$$\frac{V_D}{V_A}=\frac{h_A}{h_D}=(-1+[1+(8/Fr_A^2)]^{1/2})/2 \qquad (2)$$

It is understood that the pressure distribution at both the upstream and downstream uniform flow locations is hydrostatic.

The corresponding hodograph for this problem is shown in Fig. 2a, where the velocity $v_f(\theta)$ of the free streamline, including its value at point c, is yet unknown. The governing hodograph equation is given by[7]

$$V^2\psi_{vv}+V\psi_v+\frac{1}{\alpha^2}\psi_{\theta\theta}=0 \qquad (3)$$

where velocity v, stream function ψ, and streamline angle θ are already normalized by the upstream flow quantities and $(-\alpha)$, respectively. Values of ψ and θ now vary from zero to unity.

One now introduces a transformation of

$$q=v/v_f(\theta) \text{ and } \beta=\theta \qquad (4)$$

so that the hodograph becomes a rectangle, as shown in Fig. 2b. Equation (3) becomes, after this transformation,

$$q^2\psi_{qq}\left(1+\frac{v_f'^2}{\alpha^2V_f^2}\right)-\frac{2qv_f'}{v_f\alpha^2}\psi_{q\beta}+\frac{\psi_{\beta\beta}}{\alpha^2}+q\psi_q \cdot\left(1+\frac{2v_f'^2-v_fv_f''}{\alpha^2v_f^2}\right)=0 \qquad (5)$$

where v_f' and v_f'' indicate the first and secondary derivatives of $v_f(\theta)$, respectively. Values of ψ are completely specified on the boundary of the hodograph as indicated in Fig. 2b. For a given approaching flow condition, the values of ψ within the rectangular domain can be determined from relaxational finite difference calculations for a specific value of h_c and any arbitrary and reasonable function $v_f(\theta)$. Such calculations

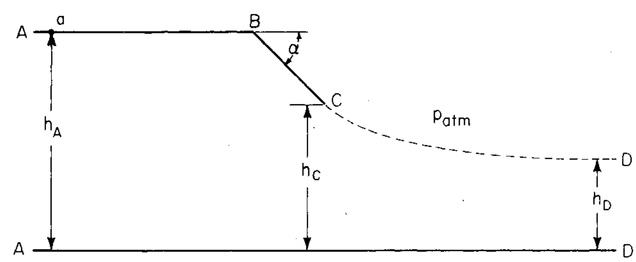

Fig. 1 Incompressible flow with gravitational effect.

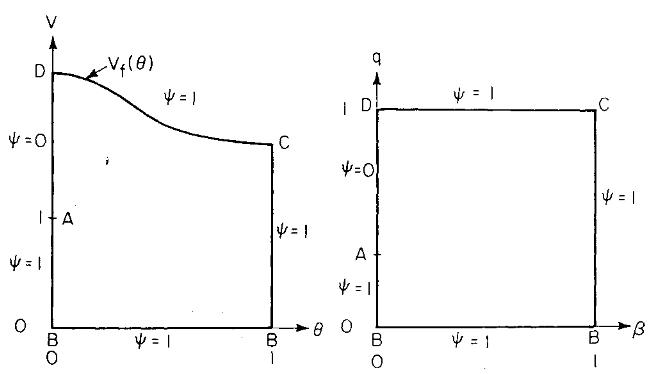

Fig. 2 The hodograph.

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*Professor, Dept. of Mechanical and Industrial Engineering. Associate Fellow AIAA.
†Graduate Research Assistant, Dept. of Mechanical and Industrial Engineering.

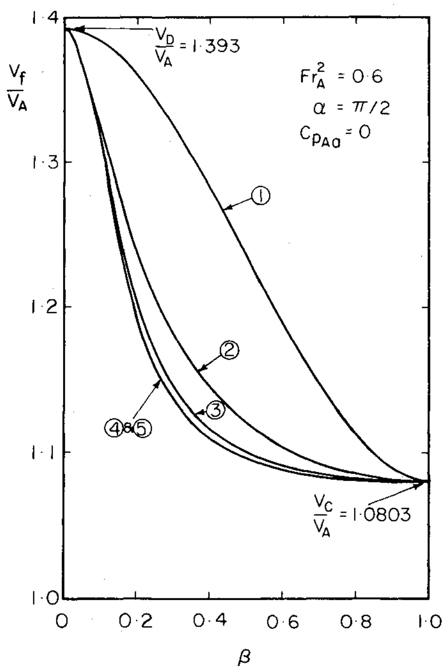

Fig. 3 Functional values of V_f at successive iterations.

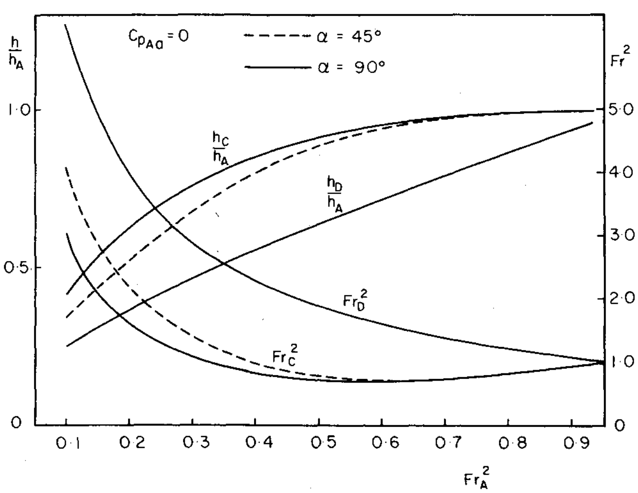

Fig. 4 Results of calculations.

may be concluded when the values of ψ within the domain are stabilized within an arbitrarily small limit (e.g., 10^{-6}). The derivative ψ_q, particularly along the free surface ($q=1$), can be established. Starting from point C in the physical plane, the profile of the free surface may be traced from integrating numerically the pair of differential equations given by

$$d\left(\frac{x}{h_A}\right)=\frac{-\cos(-\alpha\beta)\,\alpha\psi_q}{v_f}\left(1+\frac{v_f'^2}{\alpha^2 v_f^2}\right)\mathrm{d}\beta \qquad (6)$$

$$d\left(\frac{y}{h_A}\right)=\frac{-\sin(-\alpha\beta)\,\alpha\psi_q}{v_f}\left(1+\frac{v_f'^2}{\alpha^2 v_f^2}\right)\mathrm{d}\beta \qquad (7)$$

With the established new profile of the free surface, new $v_f(\theta)$ functions may be established according to

$$v_f=\sqrt{V_C^2+\frac{2}{Fr_A^2}\left(\frac{h_C}{h_A}-\frac{y_f}{h_A}\right)} \qquad (8)$$

and the foregoing calculations may be repeated until the employed $v_f(\theta)$ function also produces the same function $v_f(\theta)$ from free surface tracing. It has been found that this type of iteration is rapidly convergent. To establish $v_f(\theta)$ within the margin of accuracy of 10^{-3} at every grid point along the free surface, it rarely needs more than five iterations. It is also worthwhile to remark that the results are fairly insensitive to values of v_f''.

So far, the value of h_C has been arbitrarily selected and all kinematic relations have been satisfied. The asymptotic height h_D determined from such a free surface tracing would, in general, not agree with that obtained from the dynamic principle. It is natural to expect that the value of h_C, compatible with the given upstream (and thus downstream) condition, would produce the correct asymptotic height h_D. Again, determination of h_C is a rapidly convergent process. Only three iterations are usually needed to determine its value within an accuracy of 10^{-3}.

Figure 3 shows a set of typical successive $v_f(\theta)$ functional values for a specific flow case at various iterations. The rapidity of its convergence is apparent. Figure 4 shows the established results for various approaching flow Froude number Fr_A^2 and vanish value of $C_{p_{Aa}}$. It has been noted that for h_C/h_A larger than 0.9 ($Fr_A^2>0.45$), both h_C/h_A and Fr_C ($Fr_C=V_c/(gh_c)^{1/2}$) are sensitive and scattered within a narrow range as a result of the finite difference calculations. The curves within this range have been smoothed out in Fig. 4. It is pertinent to remark that points A and D in the hodograph are the singularities of the problem which is indeed a special feature of hodograph transformations, indicating that these conditions are reached asymptotically in the physical plane. In addition, with a uniform grid size of 0.05 and a relaxation factor of 1.6 in the establishment of ψ, completion of one case on the CYBER 175 computing system takes 2.3 s. Also, for the fixed geometrical configuration of the present problem, different supply conditions would generally lead to nonvanishing values of $C_{p_{Aa}}$.

An interesting feature of the application of hodograph transformation to the present problem is the fact that the flow conditions along the free surface provide direct influence toward the establishment of $\psi(V,\theta)$. However, these flow conditions, including the profile of the free surface, must also be coupled to $\psi(V,\theta)$ through the integrated relations given by Eqs. (6-8). From the results obtained for the present problem, it may be concluded that the hodograph transformation is a useful scheme in solving the problems where the gravitational effect is important.

Acknowledgment

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